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ORIGINAL PAPER

Random attractions and bifurcation for the classical Rayleigh–van der Pol equations with small noise

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Abstract In this paper, a classical Rayleigh–van der Pol equations with small noise is considered. Using the asymptotic expansions of the Lyapunov exponents, invariant measure and Lyapunov's direct, we investigate the stochastic bifurcation, rotation number, random limits cycle and attractor behavior of the random Rayleigh–van der Pol Equations in detail.

Keywords Rayleigh–van der Pol equations · Random attractions · Stochastic bifurcation · Random limit cycle

1 Introduction

Attractors play an important role in the study of the asymptotic behavior of dynamical systems. There is an extensive literature dealing with attractors of deterministic dynamical systems. For stochastic dynamical systems, however, comparably little progress has been made by now. Random attractors were introduced originally by Crauel and Flandoli [1], and Schmalfuß [2]. The deterministic version can be found in Hale [3]. The concept of random attractors has been extended and generalized by Arnold and Schmalfuß, Crauel, Flandoli, etc [4–13]. All references, except [6], are concerned with the Navier–Stokes equation and the methods are tailor-made for the infinite-dimensional case.

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On the other hand, stochastic bifurcation also play an important role in the study of the asymptotic behavior of random dynamical systems. Stochastic bifurcation theory studies the qualitative changes in parameterized families of random dynamical systems. Since late 1970s much work has been done on the effects of noise on the bifurcation [14–19]. However, stochastic bifurcation theory is still in its infancy [14]. There are two different approaches currently prevalent in the literature. One is the phenomenological approach favored by physicists and engineers based on the qualitative changes of stationary measure, i.e., the stationary probability density of the response. The other is the dynamical approach favored by mathematicians based on the qualitative changes of stability of invariant measures and occurrence of new invariant measures for random dynamical systems. Each approach has its advantages and the two approaches can be regarded as complementary to each other [14,17]. Recently, the stochastic Hopf bifurcation scenarios of a typical system, i.e., noisy Duffing–van der Pol oscillator, was studied numerically and a deeper insight was given [16–18,20–25].

In this paper, we present a generally applicable technique which can be used to prove the existence of stochastic Hopf bifurcation, rotation number, random limits cycle and random attractors for the random Rayleigh–van der Pol equations as follow:

$$\ddot{X} = (\alpha + \sigma_2 \xi_2(\vartheta_t \omega))\, \dot{X} - \left(\beta X^2 + \gamma \dot{X}^2\right) \dot{X} + (\mu + \sigma_1 \xi_1(\vartheta_t \omega))\, X + \sigma_3 \xi_3(\vartheta_t \omega). \tag{1.1}$$

In the random case, if $\xi_1(\vartheta_t\omega)$, $\xi_2(\vartheta_t\omega)$, $\xi_3(\vartheta_t\omega)$ are locally integrable, then Eq. (1.1) is strictly forward complete, but not backward complete, for all parameter values $\alpha, \beta, \gamma, \mu, \sigma_1, \sigma_2, \sigma_3$.

The structure of the paper is as follows. In Sect. 2, random dynamical systems are introduced, we give the definition and Lemma. In Sect. 3, we prove the existence of stochastic Hopf bifurcation, random limits cycle and random attractors for the random Rayleigh–van der Pol equations, respectively.

2 Preliminaries

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be are probability space. On Ω we define a flow θ of maps $\theta_t\colon \Omega \to \Omega$ with $t \in R$, i.e.

$$\theta_0 = id_\Omega \quad \theta_t \circ \theta_s = \theta_{t+s} \;\; s, t \in R,$$

(for brevity we write $\theta_t \circ \theta_s = \theta_t\theta_s$) such that $(t, \omega) \to \theta_t\omega$ is $\mathcal{F} \otimes \mathcal{B}(R)$-measurable and $\theta_t\mathbb{P} = \mathbb{P}$ (measure preserving). In addition $\mathbb{P}$ is assumed to be ergodic w.r.t. the flow θ. We call $\{\Omega, \mathcal{F}, \mathbb{P}, \theta_{t\, t\in R}\}$ or θ for short, a metric dynamical system.

Definition 2.1 [14] A function $\phi : T \times \Omega \times R^d \to R^d$ is called a random dynamical system (RDS) over the metric dynamical system θ, if ϕ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^{\ltimes})$, $\mathcal{B}(\mathbb{R}^{\ltimes})$-measurable and if the mappings $\phi(t, \omega) := \phi(t, \omega, \cdot) : R^d \to R^d$ form a cocycle over θ, i.e. for all $\omega \in \Omega$

$$\theta(0, \omega) = id_{R^d} \quad \phi(t+s, \omega) = \phi(t, \theta_s \omega) \circ \phi(s, \omega), \quad s, t \in \mathbb{T}.$$

Further, ϕ is called continuous RDS, if $\phi(t, \omega)$ is continuous for all $\omega \in \Omega, t \in \mathbb{T}$.

Remark 1 If Ω contains only one element ($\Omega = \{\omega\}$ and $\theta_t \omega = \omega$ for all $t \in \mathbb{R}$.) then ϕ defines a deterministic flow.

Definition 2.2 [14] A random variable $x : \Omega \to R^d$ is called a random fixed point (or stationary solution) of a given $RDS\,\phi$ if $\phi(t, \omega)x(\omega) = x(\theta_t \omega)$, for all $\omega \in \Omega, t \in \mathbb{T}$.

Definition 2.3 [12] A random compact set $D(\omega)$ maps Ω into the space of nonempty compact subset of $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ the mapping $\omega \to d(x, D(\omega))$ is measurable, where $d(x, D(\omega)) = \inf_{y \in D(\omega)} \|x - y\|$.

A nonempty family $\mathcal{D}$ consisting of the random compact sets is called inclusion closed system (IC-system), if it is maximal w.r.t. to inclusion (i.e. if $D \in \mathcal{D}$ and $D^{'}(\omega) \subset D(\omega)$ for all $\omega \in \Omega$ is a random compact set then also $D^{'} \in \mathcal{D}$).

Definition 2.4 [1] Let $\mathcal{D}$ be an IC-system. A random compact set $B \in \mathcal{D}$ is called $\mathcal{D}$-absorbing for an $RDS\,\phi$, if for only $\omega \in \Omega$ and $D \in \mathcal{D}$ there exist a $\tau_D(\omega)$ such that $\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega)$, for all $t > \tau_D(\omega)$, where $\phi(t, \omega)A := \bigcup_{y \in A} \phi(t, \omega, x)$, for all $A \subset R^n$.
Similarly we now can define a random attractor.

Definition 2.5 [14] Let $\mathcal{D}$ be an IC-system. A random compact set $A \in \mathcal{D}$ is called $\mathcal{D}$-attractor of an $RDS\,\phi$, if

(i) A is invariant, i.e. $\phi(t, \omega, A(\omega)) = A(\theta_t \omega), \quad for \ \ all \ \ t > 0, \quad \omega \in \Omega.$
(ii) A is $\mathcal{D}$-attracting, i.e. for all $\omega \in \Omega$ and $D \in \mathcal{D}$

$$\lim_{t \to +\infty} dist(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), B(\omega)) = 0,$$

where $dist(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$ is the usual Hausdorff semi-metric.

Remark 2 If $\Omega = \{\omega\}$ then the random attractors degenerate to usual deterministic attractors.

Definition 2.6 [14]

(i) A random variable $\eta : \Omega \to \mathbb{R}^+$ is called tempered if

$$\lim_{t \to \infty} \frac{\log^+ \eta(\vartheta_t \omega)}{t} = 0.$$

(ii) A non-empty random set $A(\omega)$ is called tempered if $\eta(\omega) := \sup\{\|x\|, x \in A(\omega)\}$ is tempered.

Definition 2.7 [14] Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$($\mathbb{T}$ is called 'time'), and let $((\Omega, \mathcal{F}, \mathbb{P}), (\vartheta_t)_{t\in\mathbb{T}})$ denote a metric dynamical system. That is, $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space with a flow $(\vartheta_t)_{t\in\mathbb{T}}$ on Ω, which is measurable and measure preserving, i.e. $(t, \omega) \mapsto \vartheta_t\omega$ is $\mathbb{B}(\mathbb{T}) \otimes \mathcal{F}$, $\mathcal{F}$-measurable and $\vartheta_t\mathbb{P} = \mathbb{P}\ \forall t$.

A local continuous (C^∞) random dynamical system φ over the metric dynamical system $((\Omega, \mathcal{F}, \mathbb{P}), (\vartheta_t)_{t\in\mathbb{T}})$ on $\Re^d$ is defined as a measurable mapping

$$\varphi : D \to \Re^d \quad (t, \omega, x) \mapsto \varphi(t, \omega, x)(=: (t, \omega)x),$$

where $D \in \mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(\Re^d)$, such that

(i) $\varphi(0, \omega) = id_{\Re^d}$ and locally $\varphi(t + s, \omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s, \omega)$;
(ii) $(t, x) \mapsto \varphi(t, \omega, x)$ is continuous; and
(iii) $\varphi(t, \omega) : x \mapsto \varphi(t, \omega, x)$ is a homeomorphism (C^∞ diffeomorphism) between the open sets $D(t, \omega) : \{x|(t, \omega, x) \in D\}$ and $R(t, \omega)(= D(-t, \vartheta_t\omega))$.

Lemma 2.1 [1] *Let ϕ be continuous RDS and let $\mathcal{D}$ be an IC-system, Moreover let $B \in \mathcal{D}$ be an random compact set which is $\mathcal{D}$-absorbing. Then there exist a unique $\mathcal{D}$-attractor $A \in \mathcal{D}$ for the cocycle ϕ given by*

$$A(\omega) = \bigcap_{t\geq 0} \overline{\bigcup_{t\geq\tau} \phi(\tau, \theta_{-\tau}\omega), B(\theta_{-\tau}\omega)}.$$

If $B(\omega)$ is connected then so is $A(\omega)$.

Lemma 2.2 [14] *Suppose $\eta(\vartheta_t\omega)$ is a stationary real-valued process. Whenever there exists a $\delta > 0$ such that $e^{-\delta t}|\eta(\vartheta_t\omega)| \to 0$ as $t \to \infty$, then this holds true for any $\varepsilon > 0$. The conclusion is also valid for $|\eta(\vartheta_t\omega)|$.*

Lemma 2.3 [12] *Suppose $\mathbb{E}(|\eta_1|+|\eta_2|) < \infty$ and $\mathbb{E}\eta_1 > 0$. Then random differential equation*

$$d\psi(t, \omega) = -(\eta_1(\vartheta_t\omega)\psi(t, \omega) + \eta_2(\vartheta_t\omega))dt$$

possesses the unique measurable stable stationary solution

$$r(\omega) := \int_{-\infty}^{0} e^{\int_0^s \eta_1(\vartheta_u\omega)du} \eta_2(\vartheta_s\omega)ds,$$

i.e. $\psi(t, \omega)r(\omega) = r(\vartheta_t\omega)$ for all t.

Lemma 2.4 [12] *Suppose $V : \mathbb{R} \to \mathbb{R}$ is surjective and $r(\omega)$ is measurable. Then $V^{-1}([0, r(\omega)])$ and $V^{-1}([r(\omega), \infty))$ are random sets. If, in addition, $V(U) \in \mathcal{B}(\mathbb{R}^+)$ for any open set $U \subset \mathbb{R}^d$ (e.g., if V is open), then $V^{-1}(\{r(\omega)\})$ is a random set.*

Lemma 2.5 [14] *Let φ be a continuous random dynamical system, and let $\mathcal{U}$ denote a universe of sets.*

Suppose there exists a compact set $B \in \mathcal{U}$ such that B is forward invariant ($\varphi(t,\omega)B(\omega) \subset B(\vartheta_t\omega)$ for all $t \geq 0$); B absorbs any set $D \in \mathcal{U}$; and there exists a neighborhood of B in $\mathcal{U}$.

Then

$$A(\omega) := \bigcap_{n\in\mathbb{N}} \varphi(n, \vartheta_{-n}\omega)B(\vartheta_{-n}\omega)$$

is the unique attractor for φ with domain of attraction $D(A)$ containing $\mathcal{U}$. In addition,

(i) if $B(\omega)$ is measurable, then so is $A(\omega)$ and thus A is a random attractor;
(ii) if $\omega \mapsto \varphi(t, \vartheta_{-t}\omega, x)$ is $\mathcal{F}^0_{-\infty}-$measurable and $B(\omega)$ is measurable with respect to the past, $\mathcal{F}^0_{-\infty}$, then $A(\omega)$ is $\mathcal{F}^0_{-\infty}$-measurable too; and
(iii) if $B(\omega)$ is connected, then so is $A(\omega)$.

3 Random bifurcation and attractor

In this section, we mainly consider the stochastic Hopf bifurcation, rotation number, random limits cycle and attractor for random Rayleigh–van der Pol equations.

Theorem 3.1 *Assumption $\sigma_3 = 0$, $\xi_1(\vartheta_t\omega) = \xi_2(\vartheta_t\omega) = \xi(\vartheta_t\omega)$. Then the Rayleigh–van der Pol Eq. (1.1) undergoes a Hopf bifurcation, thus a random limit cycle occurs in the stochastic system (1.1).*

Proof **Step 1:** Let $\sigma_3 = 0$, $\xi_1(\vartheta_t\omega) = \xi_2(\vartheta_t\omega) = \xi(\vartheta_t\omega)$, then Eq. (1.1) become

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ \mu & \alpha \end{pmatrix} x + \begin{pmatrix} 0 \\ -\beta x_1^2 x_2 \quad - \gamma x_2^3 \end{pmatrix} + \xi(\vartheta_t\omega) \begin{pmatrix} 0 & 0 \\ \sigma_1 & \sigma_2 \end{pmatrix} x \tag{3.1}$$

The linearized RDS Dφ is generated by the linearization of the Eq. (3.1), namely the real noise case by

$$\dot{\upsilon} = \begin{pmatrix} 0 & 1 \\ \mu & \alpha \end{pmatrix} \upsilon + \begin{pmatrix} 0 & 0 \\ -2\beta x_1 x_2 & -3\gamma x_2^2 \end{pmatrix} \upsilon + \xi(\vartheta_t\omega) \begin{pmatrix} 0 & 0 \\ \alpha_1 & \alpha_3 \end{pmatrix} \upsilon. \tag{3.2}$$

For any invariant measure ν, the trace formula gives $\lambda_1(\nu) + \lambda_2(\nu) = \alpha - 3\gamma\mathbb{E}_\nu x_2^2$. In particular, for $\nu = \delta_0$ we obtain $\lambda_1(\nu) + \lambda_2(\nu) = \alpha$.

We also need the eigenvalues of the linearization of the deterministic system $\sigma_1 = \sigma_2 = 0$ at $x = 0$,

$$\dot{\upsilon} = \begin{pmatrix} 0 & 1 \\ \mu & \alpha \end{pmatrix} \upsilon \tag{3.3}$$

which are

$$\lambda_{1,2} = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha}{4} + \mu}.$$

For $\frac{\alpha}{4} + \mu > 0$, we have two real eigenvalue, while for $\frac{\alpha}{4} + \mu < 0$ we have a pair of complex-conjugate eigenvalue

$$\lambda_{1,2} = \frac{\alpha}{2} \pm i\omega_d, \omega_d := \sqrt{\frac{\alpha}{4} + \mu},$$

where ω_d represents the "damped eigenfrequency" of the linear system

$$\ddot{X} - \alpha\dot{X} - \mu X = 0.$$

It is a general observation that noise splits deterministic multiplicities of eigenvalues. For $\alpha^2 < -4\mu$ (which we assume) and $\sigma_1 = \sigma_2 = 0$, the deterministic linear system has two complex-conjugate eigenvalues $\frac{\alpha}{2} \pm \omega_d$, which amounts to just one Lyapunov exponent $\lambda_1(0, \alpha) = \alpha/2$ with multiplicity 2. For $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$, however, the linearized SDE

$$\dot{\upsilon} = \begin{pmatrix} 0 & 1 \\ \mu & \alpha \end{pmatrix} \upsilon + \xi(\vartheta_t\omega) \begin{pmatrix} 0 & 0 \\ \alpha_1 & \alpha_3 \end{pmatrix} \upsilon \qquad (3.4)$$

has two different simple Lypunov exponents $\lambda_i(\mu, \alpha)$ which satisfy $\lambda_1(\mu, \alpha) + \lambda_2(\mu, \alpha) = \alpha$. For small σ_1, σ_2 we can use the asymptotic expansion given by Pardoux and Wihstutz [26] and others, namely when $\sigma_1 \to 0, \sigma_2 \to 0$, then Lyapunov exponent of (3.4)

$$\lambda_{1,2} = \frac{\alpha}{2} \pm \frac{\sigma_1^2 + \sigma_2^2}{8\omega_d^2} S(2\omega_d) + O\left(\max\{\sigma_1^4 + \sigma_2^4\}\right), \qquad (3.5)$$

where $S(\cdot)$ is the spectral density of $\xi(\vartheta_t\omega)$, for the white noise case where $S(u) \equiv 1$. Furthermore, the rotation number expands as

$$\rho(\mu, \alpha) = \omega_d - \frac{c(\sigma_1^2 + \sigma_2^2)}{8\omega_d^2} T(2\omega_d) + O\left(\max\{\sigma_1^4 + \sigma_2^4\}\right), \qquad (3.6)$$

where $T(\cdot)$ is the sine spectral density of $\xi(\vartheta_t\omega)$.

Consequently, at the deterministic Hopf bifurcation point $\alpha = 0$ we have for the real noise case, neglecting $O\left(\max\{\sigma_1^4 + \sigma_2^4\}\right)$ term,

$$\lambda_2(\sigma_1, \sigma_2, 0) = -\frac{\sigma_1^2 + \sigma_2^2}{8\omega_d^2} S(2\omega_d) < 0 < \lambda_1(\sigma_1, \sigma_2, 0) = \frac{\sigma_1^2 + \sigma_2^2}{8\omega_d^2} S(2\omega_d),$$

i.e. at $\alpha = 0$ the top exponent has already cross 0 and is positive. Then the Rayleigh–van der Pol Eq. (1.1) undergoes Hopf bifurcation.

Step 2: It follows from (3.5) that the top Lyapunov exponent changes sign at $\alpha_{D_1} = \frac{\sigma_1^2+\sigma_2^2}{4\mu} S(2\omega_d) + O\left(\max\{\sigma_1^4 + \sigma_2^4\}\right) < 0$ and the second Lyapunov exponent changes sign at $\alpha_{D_2} = -\frac{\sigma_1^2+\sigma_2^2}{4\mu} S(2\omega_d) + O\left(\max\{\sigma_1^4 + \sigma_2^4\}\right) > 0$.

For $\alpha < \alpha_{D_1}$, δ_0 is stable, it is the unique invariant measure, and $A_\alpha = \{0\}$ is the attractor of the RDS φ_α in $\mathbb{R}^2$.

We have a first bifurcation from δ_0 at $\alpha_{D_1} = \alpha < 0$ of a stable ergodic measure $\nu^1(\mu, \alpha)_\omega = \frac{1}{2}\left(\delta_{x^1(\omega)} + \delta_{-x^1(\omega)}\right)$, which is a convex combination of two Dirac measure, sitting on the two "boundary points" of the one-dimensional unstable manifold of $x = 0$, which is a saddle point. This situation persists for $\alpha \in (\beta_{D_1}, \alpha_{D_2})$. As shown above, $\alpha_P = 0 \in (\alpha_{D_1}, \alpha_{D_2})$, ν^1 undergoes a P–bifurcation. δ_0 and ν^1 are both Markov measure. Hence $\rho = \mathbb{E}\nu^1$ solves the Fokker–Planck equation. The attractor A_α in (the universe of tempered sets of) $\mathbb{R}^2$ is the closure of the unstable manifold of $x = 0$, the two "boundary points" supporting the measure ν^1. In the punctured plane $\mathbb{R}^2 \setminus \{0\}$, the attractor A_α^0 (in the universe of simply connected tempered sets) consist of the two-point set suppν^1.

At $\alpha = \beta_{D_2}$, we have a second bifurcation from δ_0 of a measure $\nu^2(\mu, \alpha)_\omega = \frac{1}{2}(\delta_{x^2(\omega)} + \delta_{-x^2(\omega)})$, which is again a convex combination of two Dirac measure. ν^2 is a saddle point, i.e. has a positive and negative Lyapuove exponent, while δ_0 has two positive exponent.

For $\alpha > \alpha_{D_2}$, the stable measure ν^1 is supported by the "boundary" of the unstable manifold of ν^2. The closure of this unstable manifold is an invariant "circle" around $x = 0$ ("random limit cycle") and supports both measures ν^1 and ν^2. On this "circle", we have hyperbolic dynamics (ν^1 is attracting and ν^2 is repelling), in contrast to the deterministic case, where the dynamics on the limit cycle is just rotation, and the invariant measure is unique. The interior of the "circle" is the two-dimensional unstable manifold of 0. Its closure is the attractor A in $\mathbb{R}^2$. It carries all three invariant measures. In the punctured plane $\mathbb{R}^2 \setminus \{0\}$, however, the attractor A_α^0 is the invariant "circle" , on which we have two invariant measures, in particular, the unique Markov measure ν^1. Thus the Rayleigh–van der Pol Eq. (1.1) occurs a random limit cycle. □

Theorem 3.2 *When $\sigma_2 = 0$. Let $\xi_1(\vartheta_t\omega), \xi_2(\vartheta_t\omega)$ be ergodic real noise processes with $\mathbb{E}(|\xi_1(\vartheta_t\omega)| + |\xi_2(\vartheta_t\omega)|) < \infty$. Then Eq. (1.1) generates a local $\mathcal{C}^\infty$ random dynamical system which is global forwards in time. Moreover, $\omega \mapsto \varphi(t, \vartheta_{-t}\omega, x)$ is $\mathcal{F}_{-t}^0$-measurable.*

Proof the Rayleigh–van der Pol Eq. (1.1) as an equivalent two-dimensional system of first order equation as follows:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = (\mu + \sigma_1\xi_1(\vartheta_t\omega))x - \gamma y^3 + \alpha y - \beta x^2 y + \sigma_3\xi_3(\vartheta_t\omega). \end{cases} \tag{3.7}$$

Using the canonical representation (3.7), we prove existence and uniqueness is a direct consequence of the deterministic theory and similarly for the continuity and C^∞ property, Amann [[27], Chap. II]. We prove that on any set [0, T]$\subset \Re^n$ for arbitrarily fixed sample path of the noise and initial value the solution is bounded. Since this estimate is independent of null-sets it follows that the solution is strictly complete.

Fix $T > 0, \gamma > 0, \beta > 0$ and apply the chain rule to $x^2 + y^2$, This yields

$$\begin{aligned} x^2 + y^2 &= x_0^2 + y_0^2 + 2\int_0^t xy + (\mu + \sigma_1\xi_1(\vartheta_t\omega))xy - \gamma y^4 + \alpha y^2 - \beta x^2 y^2 \\ &\quad + \sigma_3\xi_3(\vartheta_t\omega) y ds \\ &\le x_0^2 + y_0^2 + \int_0^t [1 + |\mu| + |\sigma_1|\xi_1(\vartheta_t\omega)|](x^2 + y^2) + |\alpha| y^2 \\ &\quad + |\sigma_3||\xi_3(\vartheta_t\omega)|(1 + y^2) ds \\ &\le x_0^2 + y_0^2 + \int_0^t |\sigma_3||\xi_3(\vartheta_t\omega)| ds \\ &\quad + \int_0^t [1 + |\mu| + |\alpha| + |\sigma_1||\xi_1(\vartheta_t\omega)| + |\sigma_3||\xi_3(\vartheta_t\omega)|](x^2 + y^2) ds. \end{aligned}$$

By the local integrability of $t \mapsto |\xi_1(\vartheta_t\omega)| + |\xi_2(\vartheta_t\omega)|$ the Gronwall lemma applies. Therefore, the solution exists up to time T. This holds true for each $T \in \mathbb{N}$, consequently the maximal forward solution satisfies $\tau(\omega, x) = \infty$. This result is independent of null-sets and thus the solution is strictly complete. □

Lemma 3.3 *The random dynamical system generated by the the Rayleigh–van der Pol equations, introduced in Theorem 3.2, has the additional properties*

(i) $D(t, \vartheta_{-t}\omega) = \mathbb{R}^2$ *for all* $t \ge 0$;
(ii) $R(t, \vartheta_{-t}) \downarrow E(\omega)$ *for* $t \uparrow \infty$.
$D(t, \vartheta_{-t}\omega)$ *is the domain and* $R(t, \vartheta_{-t}\omega)$ *is the range of the mapping* $\varphi(t, \vartheta_{-t}\omega)$.

Proof Property (i) follows from Theorem 3.2, because $\varphi(t, \omega, x)$ is non-explosive for any t, hence for ω. (ii) follows from the definition of $E(\omega)$ and the fact $R(t, \vartheta_{-t}) = \Re^2$ for all $t \le 0$ (which is a direct consequence of Theorem 3.2 since $R(t, \vartheta_{-t}) = D(t, \vartheta_{-t})$. The claimed inclusion $R(t, \vartheta_{-t}) \subset R(t, \vartheta_{-t})$ for $0 \le s \le t$ is implied by Definition 2.7(i). □

Theorem 3.4 *Let* $\xi_1(\vartheta_t\omega), \xi_2(\vartheta_t\omega)$ *be ergodic real noise processes with* $\mathbb{E}(|\xi_1(\vartheta_t\omega)| + |\xi_2(\vartheta_t\omega)|) < \infty$, *and assume without loss of generality that* $\sigma_1 \ge 0, \sigma_2 = 0, \sigma_3 \ge 0$. *Suppose*

(i) σ_1, σ_3 *are small enough to ensure that*

$$\delta := 1 - \frac{11}{2}\sigma_1\mathbb{E}|\xi_1(\vartheta_t\omega)| - 6\sigma_3\mathbb{E}|\xi_3(\vartheta_t\omega)| > 0.$$

Then the random dynamical system generated by the Rayleigh–van der Pol Eq. (1.1) possesses the unique parameter dependent tempered random attractor A with domain of Attraction $D(A)$ containing the universe of sets $\mathrm{Cl}(\mathcal{U})$, *generated by*

$$\mathcal{U}; = \{(D(\omega))_{\omega\in\Omega} | D(\omega) \subset \mathbb{R}^2 \text{ is a tempered random set}\}.$$

Moreover, the random attractor A is measurable with respect to the past, $\mathcal{F}_{-\infty}^{0}$.

Proof **Step 1:** Reduction to a one dimensional problem.

the Rayleigh–van der Pol Eq. (1.1) as an equivalent two-dimensional system of first order equation as follows:

$$\begin{cases} \dot{x}_1 = x_2 + \alpha x_1 - \frac{\beta}{3}x_1^3, \\ \dot{x}_2 = (\mu + \sigma_1\xi_1(\vartheta_t\omega))x_1 - \gamma x_2^3 + \sigma_3\xi_3(\vartheta_t\omega)) \end{cases} \tag{3.8}$$

where $y(t) = x_1(t)$, $\dot{y}(t) = x_2 + \alpha x_1 - \frac{\beta}{3}x_1^3$.

Define the Lyapunov function

$$V : \mathbb{R}^2 \to \mathbb{R}^+, \quad V(x_1, x_2) = \frac{7}{24}x_2^4 + x_1^2 - x_1x_2 + \frac{3}{4}x_2^2. \tag{3.9}$$

For notational simplicity only we let $((x_{1t}(\omega), x_{2t}(\omega))) := \varphi(t, \omega)$. Application of the chain rule to Eq. (3.8) implies

$$\begin{aligned} \frac{dV(x_1, x_2)}{dt} &= \frac{7}{6}x_2^3\dot{x}_2 + 2x_1\dot{x}_1 - \dot{x}_1x_2 - \dot{x}_2x_1 + \frac{3}{2}\dot{x}_2x_2 \\ &= \frac{7}{6}x_2^3\left((\mu + \sigma_1\xi_1(\vartheta_t\omega))x_1 - \gamma x_2^3 + \sigma_3\xi_2(\vartheta_t\omega)\right) \\ &\quad + (2x_1 - x_2)\left(x_2 + \alpha x_1 - \frac{\beta}{3}x_1^3\right) \\ &\quad + \left(\frac{3}{2}x_2 - x_1\right)\left((\mu + \sigma_1\xi_1(\vartheta_t\omega))x_1 - \gamma x_2^3 + \sigma_3\xi_3(\vartheta_t\omega)\right) \\ &= -\frac{7\gamma}{6}x_2^6 - \frac{3\gamma}{2}x_2^4 - \frac{2\beta}{3}x_1^4 + \left(\gamma + \frac{7\mu}{6}\right)x_2^3x_1 + \frac{\beta}{3}x_1^3x_2 \\ &\quad + (2\alpha - \mu)x_1^2 - x_2^2 + \left(2 - \alpha + \frac{3\mu}{2}\right)x_1x_2 \\ &\quad + \left(\frac{7}{6}x_1x_2^3 - x_1^2 + \frac{3}{2}x_1x_2\right)\sigma_1\xi_1(\vartheta_t\omega) \\ &\quad + \left(\frac{7}{6}x_2^3 + \frac{3}{2}x_2 - x_1\right)\sigma_3\xi_3(\vartheta_t\omega). \end{aligned} \tag{3.10}$$

The stochastic terms satisfy

$$\begin{aligned}\left|\frac{7}{6}x_1x_2^3 - x_1^2 + \frac{3}{2}x_1x_2\right| &\le \left|\frac{7}{6}x_1x_2^3\right| + \left|-x_1^2 + \frac{3}{2}x_1x_2\right| \\ &\le \frac{7}{12}(x_1^2 + x_2^6) + \frac{7}{4}x_1^2 + \frac{3}{4}x_2^2 \le \frac{11}{2}V(x_1, x_2), \\ \left|\frac{7}{6}x_2^3 + \frac{3}{2}x_2 - x_1\right| &\le \left|\frac{7}{6}x_2^3\right| + \left|\frac{3}{2}x_2 - x_1\right| \\ &\le \frac{7}{24}(4 + x_2^6) + x_1^2 + \frac{3}{2}x_2^2 + \frac{5}{2} \le 6V(x_1, x_2) + \frac{11}{3}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{dV(x_1, x_2)}{dt} &= -V(x_1, x_2) + R(x_1, x_2) \\ &\quad + \sigma_1 \frac{11}{2}\left|\xi_1(\vartheta_t\omega)\right| V(x_1, x_2) + \left(6V(x_1, x_2) + \frac{11}{3}\right)\sigma_3 \left|\xi_3(\vartheta_t\omega)\right|,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}&-\frac{7\gamma}{6}x_2^6 - \frac{36\gamma - 7}{24}x_2^4 - \frac{2\beta}{3}x_1^4 + + \frac{6\gamma + 7\mu}{6}x_2^3x_1 + \frac{\beta}{3}x_1^3x_2 \\ &+(2\alpha - \mu + 1)x_1^2 - \frac{1}{4}x_2^2\left(1 - \alpha + \frac{3\mu}{2}\right)x_1x_2.\end{aligned}$$

Note that $R(x_1, x_2) \le d(\alpha, \beta, \gamma, \mu) =: d$, where c is a positive constant independent of x_1 and x_2. Let

$$\xi(\vartheta_t\omega) := \sigma_1 \frac{11}{2}\left|\xi_1(\vartheta_t\omega)\right| + 6\sigma_3\left|\xi_3(\vartheta_t\omega)\right|, \quad \tilde{\xi}(\vartheta_t\omega) := \sigma_3 \frac{11}{3}\left|\xi_3(\vartheta_t\omega)\right|.$$

To this end we obtain the differential inequality, $\psi(t, \omega) = (x_1(\omega), x_2(\omega))$,

$$\frac{dV(\psi(t, \omega))}{dt} \le (-1 + \xi(\vartheta_t\omega))\, V(\psi(t, \omega)) + d + \tilde{\xi}(\vartheta_t\omega). \tag{3.12}$$

Step 2: Analyzing the one-dimensional problem.

Lemma 2.3 implies that the associated random differential equation possesses the dominating solution ψ, i.e.

$$V(\varphi(t, \omega, (x_1, x_2))) \le \psi(t, \omega, V(x_1, x_2)), \tag{3.13}$$

and ψ possesses the unique measurable stationary stable solution

$$r(\omega) := \int_{-\infty}^{0} e^{s - \int_0^s \xi_1(\vartheta_u \omega) du} \left(d + \tilde{\xi}(\vartheta_s \omega)\right) ds. \tag{3.14}$$

$r(\omega)$ is called “stationary solution”, because $\psi(t,\omega)r(\omega) = r(\vartheta_t \omega)$ for all t. In particular, $r(\omega)$ and $(\vartheta_t \omega)$ have the same distribution for all t.

One has $\psi(t, \vartheta_{-t}\omega)x(\vartheta_{-t}\omega) \to r(\omega)$ as $t \to \infty$ for any initial value $x(\omega) \in \mathbb{R}^+$ such that $e^{-\delta t} x(\vartheta_{-t}\omega) \to 0$. Thus we may define the universe of sets

$$\mathcal{U}_1 := \{(I(\omega))_{\omega \in \Omega} | I(\omega) \subset \mathbb{R}^+ \text{ is a tempered random set}\}.$$

By definition of the universe $\mathcal{U}_1$ the random variable $\eta_1(\omega) := \sup\{x | x \in I(\omega)\}$ satisfies $\lim_{t\to\infty} \log^+ \eta_1(\vartheta_t \omega)/t = 0$. Thus, we obtain the needed condition $\lim_{t\to\infty} e^{-\delta t} \eta_1(\vartheta_{-t}\omega) = 0$.

$\mathcal{U}_1$ is closed under inclusion and contains in particular any deterministic point $\{x\} \subset \mathbb{R}^+$, i.e. $\Omega \times \{x\} \in \mathcal{U}_1$.

The random set $[0, (1+\varepsilon)r(\omega)]$ ($\varepsilon > 0$ is arbitrarily fixed) is forward invariant and absorbing for $\psi(t,\omega)$ with respect to the universe $\mathcal{U}_1$.

The forward invariance can be seen as follows. Since $r(\omega) \geq 0$, $\psi(t,\omega,0) \geq 0$ (by non-negativity of V), and $x < y \Rightarrow \psi(t,\omega,x) < \psi(t,\omega,y)$, it suffices to show that

$$\psi(t, \vartheta_{-t}\omega, (1+\varepsilon)r(\vartheta_{-t}\omega)) \leq (1+\varepsilon)r(\omega).$$

By Lemma 2.3 the left-hand side is equal to

$$e^{-t + \int_0^t \xi_1(\vartheta_{s-t}\omega) ds} \left((1+\varepsilon)r(\vartheta_{-t}\omega) + \int_0^t e^{s - \int_0^s \xi(\vartheta_{u-t}\omega) du} \left(d + \tilde{\xi}(\vartheta_{s-t}\omega)\right) ds \right).$$

Since $d + \tilde{\xi}(\omega) \geq 0$ (which is essential) this quantity is bounded by

$$e^{-t + \int_0^t \xi_1(\vartheta_{s-t}\omega) ds} (1+\varepsilon) \int_0^t e^{s - \int_0^s \xi(\vartheta_{u-t}\omega) du} \left(d + \tilde{\xi}(\vartheta_{s-t}\omega)\right) ds$$

$$(1+\varepsilon) \int_0^t e^{s - t - \int_0^{s-t} \xi(\vartheta_u \omega) du} \left(d + \tilde{\xi}(\vartheta_{s-t}\omega)\right) ds.$$

Substituting $v = s - t$ we obtain that this term is equal to $(1+\varepsilon)r(\omega)$.

The absorption property follows immediately from the definition of $\mathcal{U}_1$. Notice, $[0, r(\omega)]$ is attracting but not absorbing.

Finally, note that $[0, (1+\varepsilon)r(\omega)]$ is an element of $\mathcal{U}_1$. This holds true because

$$e^{-\delta t} r(\vartheta_{-t}\omega) \to 0 \text{ as } t \to \infty$$

and hence, by stationarity, $e^{-\varepsilon t} r(\vartheta_{-t}\omega) \to 0$ for any $\varepsilon > 0$, see Lemma 2.2

Step 3: "Lift" to the original coordinates.

Define

$$B(\omega) := V^{-1}([0, (1+\varepsilon)r(\omega)]) \subset \mathbf{R}^2, \text{ with an arbitrarily fixed } \varepsilon > 0.$$

This a non-empty compact set by the surjectivity and continuity of V, and the fact that pre-images of bounded sets are bounded under V.

The family $(B(\omega))_{\omega\in\Omega}$ of compact sets satisfies the assumption of Lemma 2.5. To prove this we have to show (a) measurability; (b) forward invariance, i.e. $\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subset B(\omega)$ for any $t \geq 0$; (c) absorption of any set $D \in \mathcal{U}$, where the universe of sets $\mathcal{U}$ is defined in the Lemma to be proved; and (d) existence of a neighborhood of B in $\mathcal{U}$.

V is surjective and therefore measurability follows from Lemma 2.4.

The forward invariance can be seen as follows. We obtain

$$\begin{aligned} &\psi(t, \vartheta_{-t}\omega)[0, (1+\varepsilon)r(\vartheta_{-t}\omega)] \subset [0, (1+\varepsilon)r(\omega)] \\ &\Leftrightarrow \quad \psi(t, \vartheta_{-t}\omega)V(B(\vartheta_{-t}\omega)) \subset V(B(\omega)) \\ &\Rightarrow \quad V(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega)) \subset V(B(\omega)) \end{aligned}$$

where we have used surjectivity of V, the domination property (3.13), and nonnegativity of V. Now (b) follows resdly

$$\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subset V^{-1}(V(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega))) \subset V^{-1}(V(B(\omega))).$$

Property (c) is a consequence of the domination property of ψ, the fact that $\psi(t, \vartheta_{-t}\omega)x(\vartheta_{-t}\omega) \to r(\omega)$ when $t \to \infty$ for $x(\cdot) \in \mathcal{U}_1$, and the surjectivity of V.

For any $D \in \mathcal{U}$ it is $V(D) \in \mathcal{U}_1$. This follows from the fact that $x^4 + y^2 \leq \|(x, y)\|^4 + 1$, $V(x, y) \leq x^4 + y^2 + const.$, and the random variable $\sup\{\|(x, y)\|^4 + const. | (x, y) \in D(\omega)\}$ grows sub-exponentially fast by the definition of $\mathcal{U}$.

Since $D \in \mathcal{U} \Rightarrow V(D) \in \mathcal{U}_1$ and $[0, (1+)r(\omega)]$ absorbs any set in $\mathcal{U}_1$ there exists a $t(\omega, V(D))(=: t(\omega, D))$ such that for any $t \geq t(\omega, D)$

$$V(\varphi(t, \vartheta_{-t}\omega)D(\vartheta_{-t}\omega)) \subset \varphi(t, \vartheta_{-t}\omega)V(D(\vartheta_{-t}\omega)) \subset [0, (1+)r(\omega)] = V(B(\omega)).$$

As in the proof of property (b), we take pre-images with respect to V^{-1} of the first and the last term. This gives (c).

Finally, property (d) follows from the fact that there exists a positive random variable $d(\omega)$ such that $B(\omega) \subset B_0(d(\omega)) := \{x \in \mathbb{R} | \|x\| \leq d(\omega)\} \in \mathcal{U}$. Therefore, $(B_0(ad(\omega)))_{\omega\in\Omega}$ for any $a > 0$, which is a neighborhood of B for an arbitrarity fixed $a > 1$.

Application of Lemma 2.5 finishes the proof. Note that the attractor $A(\omega)$ is measurable with respect to the past, $\mathcal{F}_{-\infty}^{0}$, because $B(\omega)$ has this measurability by construction. □

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